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ON THE HEREDITY OF WEAK COMPACT GENERATING

BY

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ABSTRACT

If X is a Banach space such that X, X^* are subspaces of Banach spaces generated by weakly-compact sets, then X is also generated by a weakly-compact set and admits an equivalent Fréchet smooth norm.

1. Introduction

It was shown by H. P. Rosenthal [10] that the property of being weakly-compactly generated (WCG) is not hereditary. Neverthless, in the class of Banach spaces having an equivalent Fréchet smooth norm, WCG property is already hereditary [2, 6]. Furthermore, W. B. Johnson and J. Lindenstrauss constructed a non-WCG Banach space X with X^* WCG [9]. However, if X is a subspace of a WCG space and X^* is WCG, then X is WCG (cf. [9, 7]).

The purpose of this note is to show the result stated in the Abstract. It extends also the result of [4] which states that if X, X^* are both WCG, then X has an equivalent Fréchet smooth norm.

2. Notations and definitions

A real Banach space X is generated by a weakly-compact set $K \subset X$ if $X = \overline{\mathrm{sp}} K$, where $\overline{\mathrm{sp}}\, K$ means the closed linear hull of K. A system $\{(x_\alpha, f_\alpha), \alpha \in \Gamma\}$, $x_\alpha \in X$, $f_\alpha \in X^*$ is a shrinking Markuševič basis of X if $f_\alpha(x_\beta) = \delta_{\alpha\beta}$, $\overline{\mathrm{sp}}\{x_\alpha\} = X$, $\overline{\mathrm{sp}}\{f_\alpha\} = X^*$. A norm $|\cdot|$ of a Banach space X is locally, uniformly rotund (LUR) if whenever x, $x_n \in X$, $|x| = |x_n| = 1$, $\lim |x + x_n| = 2$, then $\lim |x - x_n| = 0$. If Y, Z are closed subspaces of a Banach space X such that $X = \overline{Y + Z}$, $Y \cap Z = \{0\}$, then Y, Z are called quasicomplements to each other in X. Unless stated otherwise, by a subspace we will mean a closed subspace. If K is an absolutely convex set in a linear space, then $|\cdot|_K$ denotes the seminorm on sp K (the linear hull of K) given by K and the topology of this seminorm is refered to as the K-topology. T/Z denotes the restriction of the map T to Z. If

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$K \subset X$, then $K^0 \subset X^*$ is the polar of K in X^*. By Re, we will mean the restriction to X, where $X \subset U$. Thus, Re: $U^* \rightarrow X^*$.

3. Statement of the results

Before proceeding to formulation of the results of this paper, we recall that if the dual norm of X^* is LUR, then the norm of X is Fréchet differentiable (on $X \setminus \{0\}$) by the result of R. Lovaglia (cf. [3]). Furthermore, if $\{(x_\alpha, f_\alpha)\}$ is a shrinking Markuševič basis of a Banach space X, then easily, $\cup x_\alpha \cup \{0\}$ is a weakly compact set generating X, if we suppose that $\{x_\alpha\}$ is bounded. Thus, we may state our main result as:

THEOREM. *If X is a Banach space such that X, X^* are subspaces of WCG Banach spaces U, V respectively, then X admits a shrinking Markuševič basis.*

COROLLARY 1. *Under the assumptions of the Theorem, X has an equivalent norm whose dual on X^* is LUR.*

COROLLARY 2. *Under the assumptions of the Theorem, and if $Y \subset X^*$ is a norm closed subspace of X^*, then Y has a Markuševič basis with dual coefficients from X. Moreover, if Y is total on X, then X has a Markuševič basis with dual coefficients from Y. Also, Y has a w^* closed quasicomplement in X^*.*

4. Proof of the results

The proof of the Theorem is built up in a series of lemmas and is based on the construction of suitable resolutions of identity, which was developed by J. Lindenstrauss and D. Amir [1]. Here we use also some arguments from [4, 5], namely the method for constructing w^* continuous projections invariant to some subspaces.

LEMMA 1. *Let X, U, V be as in the Theorem and let K_1, K_2 be absolutely-convex, weakly compact sets such that $U = \overline{\text{sp}}\, K_1$, $V = \overline{\text{sp}}\, K_2$. Let $B \subset X^*$ and $W \subset \overline{\text{sp}} K_2$ be finite dimensional subspaces. Let $\{b_i\}$ be a basis of B and let $\tilde{b}_i \in U^*$ be such that* Re $\tilde{b}_i = b_i$. *Put $\tilde{B} = \text{sp}\ \{\tilde{b}_i\}$. Then, there are linear projections P on V and $\tilde{P}$ on U^* with the properties:*

(i) *P is continuous on V;*

(ii) *$PW \subset W$ and $P/\text{sp}\ K_2$ is K_2-continuous;*

(iii) *P/X^* is w^* continuous;*

(iv) *P is onto B; and*

(i′) *$\tilde{P}$ is w^* continuous (on U^*)*

(ii′) $\tilde{P}X^{\perp} \subset X^{\perp}$

(iii′) $\operatorname{Re} \tilde{P}u^* = P \operatorname{Re} u^*$ *for all* $u^* \in U^*$

(iv′) $\tilde{P}$ *is onto* $\tilde{B}$.

PROOF. Let τ be a locally-convex topology on V which is weaker than the norm topology on V, the K_2-topology on sp K_2 and the w^* topology on X. Let $\{a_i\}$ be a basis of $B \cap W$.
Let us complete $\{a_i\}$ by systems $\{c_i\}$ and $\{d_i\}$ to bases of B and W respectively. Then $\{a_i\} \cup \{c_i\} \cup \{d_i\}$ is a basis of $\operatorname{sp}(B \cup W)$. Define on $\operatorname{sp}(B \cup W)$ a projection P by

$$P(\Sigma\alpha_i a_i + \Sigma\beta_i c_i + \Sigma\gamma_i d_i) = \Sigma\alpha_i a_i + \Sigma\beta_i c_i \ .$$

Now let us extend P to the whole V by the Hahn-Banach theorem used for the τ topology.

This construction produces a linear projection P on V with the properties (i)–(vi). If b_i is the basis of B from Lemma 1, then there are $x_1, \cdots x_n \in X (n = \dim B)$, such that if $x^* \in X^*$, then

$$Px^* = \Sigma x^*(x_i) b_i.$$

If we define on U

$$\tilde{P}u^* = \Sigma u^*(x_i)\tilde{b_i},$$

we have the second desired projection. The following definition will be suitable.

DEFINITION. *Let* X, U, V, K_2, $B = \operatorname{sp}\{b_i\}$, $\tilde{B} = \operatorname{sp}\{\tilde{b_i}\}$ *be as in Lemma* 1 *and let* $Z \subset V$ *and* $\tilde{Z} \subset U^*$ *be finite dimensional subspaces,* $B \subset Z$, $\dim Z/B = n$. *Put* $W = (\operatorname{sp} K_2) \cap Z$. *We say that* Z, $\tilde{Z}$ *form a suitable pair of subspaces if there are projections* P, $\tilde{P}$ *with all the properties from Lemma* 1 *and a basis* $(z_1, \cdots, z_n)$ *of* $(I - P)Z$ *and a basis* $(\tilde{z}_1, \cdots, \tilde{z}_b)$ *of* $\tilde{Z}$, *such that*

(1) $(z_1, \cdots, z_a)$ *is a basis of* $(I - P)Z \cap \operatorname{sp} K_2 \cap X^*$;

(2) $(z_1, \cdots z_b)$ *is a basis of* $(I - P)Z \cap X^*$;

(3) $(z_1, \cdots z_a, z_{b+1}, \cdots, z_c)$ *is a basis of* $(I - P)\, Z \cap \operatorname{sp} K_2$;

(4) $\operatorname{Re}\, \tilde{z}_i = z_i$ *for* $i = 1, \cdots, b$.

LEMMA 2. *Let* X, U, V, K_1, K_2, $B = \operatorname{sp}\{b_i\} \subset X^*$, $\tilde{B} = \operatorname{sp}\{\tilde{b}\} \subset U^*$ *be as in Lemma* 1. *Furthermore, let* $f_1, \cdots, f_k \in X$, *and let* m, n *be positive integers. Then* *there are $\aleph_0$-dimensional subspace $C \subset V$ and $\aleph_0$-dimensional subspace $\tilde{C} \subset U^*$ such that for any $\varepsilon > 0$ and any suitable pair of subspaces $Z \subset V$, $\tilde{Z} \subset U^*$ with $B \subset Z$, $\dim Z/B = n$ and for any subspace $F \subset X^\perp \subset U^*$, $\dim F = m$, there exist linear operators $T = T(Z): Z \to C$ and $\tilde{T} = \tilde{T}(\tilde{Z}, F): \tilde{B} \oplus \tilde{Z} \oplus F \to \tilde{C}$ such that;*

(i) $|T| \leqq 1 + \varepsilon$;

(ii) $T(Z \cap \operatorname{sp} K_2) \subset \operatorname{sp} K_2$ and $|T/Z \cap \operatorname{sp} K_2|_{K_2} \leqq 1 + \varepsilon$;

(iii) $T(Z \cap X^*) \subset X^*$;

(iv) $|f_i(Tx^*) - f_i(x^*)| \leqq \varepsilon |x^*|$ for any $x^* \in X^* \cap Z$ and $i = 1, \cdots, k$;

(v) $Tb = b$ *for* $b \in B$;

(i') $|\tilde{T}| \leqq 1 + \varepsilon$ *and* $|\tilde{T}|_{K_1^0} = 1 + \varepsilon$;

(ii') $\tilde{T}F \subset X^\perp$;

(iii') $\operatorname{Re}\tilde{T}u^* = T\operatorname{Re}u^*$ *for all* $u^* \in \tilde{B} \oplus \tilde{Z} \oplus F$.

Proof. Let r be a positive integer. Let $\tilde{b}_1, \cdots, \tilde{b}_p \in \tilde{B}$ be so that

1) if $\tilde{b} \in \tilde{B}$, $|\tilde{b}| \leqq r$, then there is $h \in \{1, \cdots, p\}$ such that $|\tilde{b} - \tilde{b}_h| \leqq r^{-1}$;

2) if $\tilde{b} \in B$, $|\tilde{b}|_{K^0} \leqq r$, then there is $h \in \{1, \cdots, p\}$ such that $|\tilde{b} - \tilde{b}_h|_{K_1^0} \leqq r^{-1}$;

3) if $\tilde{b} \in \tilde{B}$, $|\operatorname{Re}\tilde{b}| \leqq r$, then there is $h \in \{1, \cdots, p\}$ such that $|\operatorname{Re}\tilde{b} - \operatorname{Re}\tilde{b}_h| \leqq r^{-1}$; and

4) if $\tilde{b} \in \tilde{B}$, $|\operatorname{Re}\tilde{b}| \leqq r$, $\operatorname{Re}\tilde{b} \in \operatorname{sp} K_2$, then there is $h \in \{1, \cdots, p\}$ such that $\operatorname{Re}\tilde{b}_h \in \operatorname{sp} K_2$ and $|\operatorname{Re}\tilde{b} - \operatorname{Re}\tilde{b}_h|_{K_2} \leqq r^{-1}$.

Assume that $\tilde{b}_1, \cdots, \tilde{b}_{p'}$ $(p' \leqq p)$ are exactly those $\tilde{b}_1, \cdots, \tilde{b}_p$ for which $\operatorname{Re}\tilde{b}_h \in \operatorname{sp} K_2$. Consider the Euclidean space R^{n+m} with the norm $|\lambda| = \Sigma|\lambda_i|$. Choose elements $\lambda^1, \cdots, \lambda^q$ of the unit sphere $S^{n+m} = \{\lambda \in R^{n+m}; |\lambda| = 1\}$ such that for any $\lambda \in S^{n+m}$ there is $j \in \{1, \cdots, q\}$ such that $|\lambda - \lambda^j| \leqq r^{-1}$.

If a, b, c are integers, $a, b, c \in \langle 1, n \rangle$, let us consider the following set of $(n + m + b)$-tuples: $H = \{(x_1 \cdots, x_n, \tilde{x}_1, \cdots, \tilde{x}_b, \tilde{x}_{n+1}, \cdots, \tilde{x}_{n+m})\}$, where $x_1, \cdots, x_a \in X^* \cap \operatorname{sp} K_2$, $x_{a+1}, \cdots, x_b \in X^*$, $x_{b+1}, \cdots, x_c \in \operatorname{sp} K_2, x_{c+1}, \cdots, x_n \in V$, $\tilde{x}_1, \cdots, \tilde{x}_b$ are continuous extensions of $x_1, \cdots, x_b$ to U, $\tilde{x}_{n+1}, \cdots, \tilde{x}_{n+m} \in F$.

Now if the integers $r, a, b, c, (1 \leqq a \leqq b \leqq c \leqq n)$ are given, let us define on H the following $Q = n + (a + c - b) + pq + p'q + kb + 2(b + m) + 2pq$ real valued functions:

$$|x_i| \quad (i = 1, \cdots, n), \quad |x_i|_{K_2} \quad (i = 1, \cdots a, b + 1, \cdots, c);$$

$$\left| \operatorname{Re}\tilde{b}_h + \sum_{i=1}^{n} \lambda_i^j x_i \right| \quad (h = 1, \cdots, p;\ j = 1, \cdots, q);$$

$$\left| \operatorname{Re}\tilde{b}_h + \left(\sum_{i=1}^{a} + \sum_{i=b+1}^{c} \right) \lambda_i^j x_i \right|_{K_2} \quad (h = 1, \cdots, p',\ j = 1, \cdots q);$$

$f_l(x_i) \quad (l = 1,\cdots,k;\ \ i = 1,\cdots,b);$

$|\tilde{x}_i| \quad (i = 1,\cdots,b,n+1,\cdots,n+m);\quad |x_i|_{K_1^0} \quad (i = 1,\cdots,b,n+1,\cdots,n+m);$

$$\left|\tilde{b}_h + \sum_{i=1}^{b} \lambda_i^j \tilde{x}_i + \sum_{i=n+1}^{n+m} \lambda_i^j \tilde{x}_i\right| \quad (h = 1,\cdots,p,\ j = 1,\cdots,q);$$

$$\left|\tilde{b}_h + \sum_{i=1}^{b} \lambda_i^j \tilde{x}_i + \sum_{i=n+1}^{n+m} \lambda_i^j \tilde{x}_i\right|_{K_1^0} \quad (h = 1,\cdots,p, j = 1,\cdots,q).$$

These functions may be regarded as a function ϕ on H to R^Q.

Taking in R^Q the metric ρ of maximal coordinate distance, we see that we may choose for every fixed r, a, b, c a sequence $\{h^t\} \subset H$ such that $\phi(\{h^t\})$ is ρ-dense in $\rho(H)$. Let h_i^t be the i^{th} coordinate of h^t. Now let C be a subspace spanned by B and by h_i^t, $i = 1,\cdots,n$, $t = 1.2,\cdots$ and this for all $r = 1,2,\cdots$ and all a, b, c with $1 \leqq a \leqq b \leqq c \leqq n$. Similarly let $\tilde{C}$ be the subspace spanned by $\tilde{B}$ and by h_i^t $i = n+1,\cdots,n+b, 2n+1,\cdots,2n+m$, $t = 1,2,\cdots$ and also for all admissible r, a, b, c.

Suppose now we are given an $\varepsilon > 0$, a suitable pair of subspaces $Z \subset V$, $\tilde{Z} \subset U^*$ with $Z \supset B$, $\dim Z/B = n$ and a subspace $F \subset X^\perp$, with $\dim F = m$. Let P, $\tilde{P}, z_1,\cdots,z_a,z_{a+1},\cdots,z_b,z_{b+1},\cdots,z_c,z_{c+1},\cdots,z_n \tilde{z}_1,\cdots,\tilde{z}_b$ be from the Definition of $Z, \tilde{Z}$.

Furthermore, let $\tilde{z}_{n+1},\cdots,\tilde{z}_{n+m}$ be a basis of F. We may suppose that

$$\left|\sum_{i=1}^{n} \lambda_i z_i\right| \geqq |(\lambda_1,\cdots,\lambda_n)|,$$

$$\left|\left(\sum_{i=1}^{a} + \sum_{i=b+1}^{c}\right)\lambda_i z_i\right|_{K_2} \geqq |(\lambda_1,\cdots,\lambda_a,\lambda_{b+1},\cdots,\lambda_c)|,$$

$$\left|\left(\sum_{i=1}^{b} + \sum_{i=n+1}^{n+m}\right)\lambda_i \tilde{z}_i\right| \geqq |(\lambda_1,\cdots,\lambda_b,\lambda_{n+1},\cdots,\lambda_{n+m})|,$$

$$\left|\left(\sum_{i=1}^{b} + \sum_{i=n+1}^{n+m}\right)\lambda_i \tilde{z}_i\right|_{K_1^0} \geqq \left|(\lambda_1,\cdots,\lambda_b,\lambda_{n+1},\cdots,\lambda_{n+m})\right|,$$

for any λ_i's (it suffices to multiply all z_i, $\tilde{z}_i$ by a sufficiently large number).

Let s be so that $|z_i| \leqq s$ $(i = 1,\cdots,n)$, $|z_i|_{K_2} \leqq s$ $(i = 1,\cdots,a,b+1,\cdots,c)$, $|\tilde{z}_i| \leqq s$ $(i = 1,\cdots,b,n+1,\cdots,n+m)$, $|\tilde{z}_i|_{K_1^0} \leqq s$ $(i = 1,\cdots,b,n+1,\cdots,n+m)$. Let moreover

$$|P/Z| \leqq K, \quad |P/(Z \cap \operatorname{sp} K_2)|_{K_2} \leqq K, \quad |\tilde{P}/(\tilde{B} \oplus \tilde{Z} \oplus F)| \leqq K,$$
$$|\tilde{P}/(\tilde{B} \oplus \tilde{Z} + F)|_{K_1^0} \leqq K$$

(observe that no continuity of projections P, $\tilde{P}$ on the whole spaces is needed). Let $M > 1$ be so that $M > 6(1+K)^{-1}$ and choose an integer r so that $2s+1 \leqq \varepsilon(r-s)$ and $r^{-1}s \leqq M^{-1}$. We will fix these r, a, b, c in the rest of the proof (a, b, c are from the Definition of Z, $\tilde{Z}$).

Let

$$x = (x_1, \cdots, x_n, \tilde{x}_1, \cdots, \tilde{x}_b, \tilde{x}_{n+1}, \cdots, \tilde{x}_{n+m}) \in \{h^t\}$$

be such that $\rho(\phi(x), \phi(z)) \leqq M^{-1}$, where

$$z = (z_1, \cdots, z_n, \tilde{z}_1, \cdots, \tilde{z}_b, \tilde{z}_{n+1}, \cdots, \tilde{z}_{n+m}).$$

Define on Z

$$T\left(b + \sum_{i=1}^{n} \lambda_i z_i\right) = b + \sum_{i=1}^{n} \lambda_i x_i \quad (b \in B),$$

and on $\tilde{B} \oplus \tilde{Z} \oplus F$

$$\tilde{T}\left(\tilde{b} + \left(\sum_{i=1}^{b} + \sum_{i=n+1}^{n+m}\right) \lambda_i \tilde{z}_i\right) = b + \left(\sum_{i=1}^{b} + \sum_{i=n+1}^{n+m}\right) \lambda_i \tilde{x}_i \qquad (\tilde{b} \in \tilde{B}).$$

We prove that $\tilde{T}F \subset X^{\perp}$. If

$$f = \tilde{b} + \left(\sum_{i=1}^{b} + \sum_{i=n+1}^{n+m}\right) \lambda_i \tilde{z}_i \in F,$$

then

$$\operatorname{Re}\left(\tilde{b} + \sum_{i=1}^{b} \lambda_i \tilde{z}_i\right) = 0 \quad \text{on } X.$$

Thus $\lambda_i = 0$ for $i = 1, \cdots, b$ and $\tilde{b} = 0$ because $z_i \in (I-P)Z$. Therefore, we have

$$\tilde{T}f = \sum_{i=n+1}^{n+m} \lambda_i \tilde{x}_i \in X^{\perp}.$$

Furthermore, if

$$f = \tilde{b} + \sum_{i=1}^{b} \lambda_i \tilde{z}_i + \sum_{i=n+1}^{n+m} \lambda_i \tilde{z}_i \in \tilde{B} \oplus \tilde{Z} \oplus F,$$

then

$$\operatorname{Re} Tf = \operatorname{Re}\left(\tilde{b} + \sum_{i=1}^{b} \lambda_i \tilde{x}_i + \sum_{i=n+1}^{n+m} \lambda_i \tilde{x}_i\right) = \operatorname{Re} \tilde{b} + \sum_{i=1}^{b} \lambda_i x_i = T \operatorname{Re} f.$$

The other properties of the operators T, $\tilde{T}$ are derived similarly as in [5] (proof of Lemma 2).

LEMMA 3. *Let X, U, V, K_1, K_2, $B = \mathrm{sp}\{b_i\} \subset X^*$, $\tilde{B} = \mathrm{sp}\{\tilde{b}_i\} \subset U^*$ be as in Lemma 1, let $\aleph$ be an infinite cardinal number, and let $B \subset X^*$ and $D \subset X$ be subspaces with* dens $B \leqq \aleph$, dens $D \leqq \aleph$. *Then there are linear operators P: $V \to V$ and T: $U^* \to U^*$ such that*

(i) $|P| = 1$;
(ii) $P(\mathrm{sp}\, K_2) \subset \mathrm{sp}\, K_2$ *and* $|P/\mathrm{sp}\, K_2|_{K_2} = 1$;
(iii) $PX^* \subset X^*$;
(iv) $(P/X^*)^* f = f$ *for* $f \in D$;
(v) $Pb = b$ *for* $b \in B$;
(vi) P/X^* *is* $w^* - w^*$ *continuous*;
(vii) P/X^* *is a projection*; *and*
(i′) $|T| = |T|_{K_1^0} = 1$;
(ii′) $TX^\perp \subset X^\perp$;
(iii′) $\mathrm{Re}\, Tu^* = P\, \mathrm{Re}\, u^*$ *for all* $u^* \in U^*$.

PROOF. By induction on $\aleph$. If $\aleph = \aleph_0$, we first assume that $\dim B < \infty$ and prove that then there exist linear operators P: $V \to V$ and T: $U^* \to U^*$ having all properties stated in Lemma 3 except that P/X^* is a projection (see Lemma 3 in [1]). For this purpose, let us consider the net $\{(Z, \tilde{Z}, F)\}$, where $B \subset Z \subset V$, $\tilde{Z} \subset U^*, Z, \tilde{Z}$ form a suitable pair of subspaces (see Definition), $F \subset X^\perp$, $\dim F < \infty$. Ordering is given by: $(Z_1, \tilde{Z}, F_1) < (Z_2, \tilde{Z}_2, F_2)$ iff $Z_1 \subset Z_2$ and $\tilde{B} \oplus \tilde{Z}_1 \oplus F_1 \subset \tilde{B} \oplus \tilde{Z}_2 \oplus F_2$. It is easy to see that the Z's and $(\tilde{B} \oplus \tilde{Z} \oplus F)$'s ($Z$, $\tilde{Z}$, F are from this net) exhaust all V and U^* respectively.

Let $\{f_i\}$ be a dense sequence in the unit sphere of D. It is easy to see by Lemma 2 that there are $\aleph_0$-dimensional subspaces $C \subset V$, $\tilde{C} \subset U^*$ such that for any $(Z, \tilde{Z}, F) \in \{(Z, \tilde{Z}, F)\}$, there exist linear operators $T(Z)$: $Z \to C$, $\tilde{T}(\tilde{Z}, F)$: $\tilde{B} \oplus \tilde{Z} \oplus F \to \tilde{C}$ with the properties from Lemma 2 for $\varepsilon = (\dim Z/B)^{-1}$ and $k = \dim Z/B$. If U_1 is the unit ball of U^*, then using Tichonov theorem for $(2U_1)^{U_1}$ and for $(2K_2)^{K_2}$ in the w^* and w topologies respectively, we see, similarly as in [1] that there is a subnet $\{(Z_\nu, \tilde{Z}_\nu, F_\nu)\}$ of $\{(Z, \tilde{Z}, F)\}$ such that the operators $T(Z_\nu)$, $\tilde{T}(\tilde{Z}_\nu, F_\nu)$ converge to the operators P, T in the weak and w^* operator topologies respectively. We extend P continuosly on the whole V. Easily P and T have all the properties (i)–(v) and (i′)–(iii′) from the statement. Now (iii′) implies that $(P/X^*)^* x = T^* x$ for $x \in X$ and (ii′) implies $T^* X \subset X$. Thus $(P/X^*)^* X \subset X$ which is (vi). The first statement of this proof is proved.

Then we prove the whole Lemma 3 by use of arguments of [1] (Lemma 4) and those given above.

LEMMA 4. *Let X be as in the Theorem and let μ be the first ordinal of cardinality* dens X^*. *Then there is a long sequence of projections $\{P_\alpha : 0 \leqq \alpha \leqq \mu\}$ on X^*, $P_0 = 0$, P_μ = identity, $P_\alpha P_\beta = P_\beta P_\alpha = P_\beta$ if $\beta < \alpha$, $\|P_\alpha\| = 1$ for $\alpha > 0$,* dens $P_\alpha X^* = \bar{\bar{\alpha}}$ *for $\alpha \geqq \omega$, $P_\alpha x^*$ is norm-continuous on ordinals for any fixed $x^* \in X^*$ and P_α's are $w^* - w^*$ continuous.*

PROOF. Using Lemma 3, the arguments of [1, 5] and above.

PROOF OF THE THEOREM. By induction on dens X^*. If dens $X^* = \aleph_0$, then X has a shrinking Markuševič basis (see [3.8]). If the Theorem holds for any space Y with dens $Y^* < \aleph$ and if dens $X^* = \aleph$, then $(P_{\alpha+1} - P_\alpha)X$ has a shrinking Markuševič basis for any $\alpha < \mu$ (the projections P_α on X are the preduals of the projections from Lemma 4). This follows from the fact that $(P_{\alpha+1} - P_\alpha)X \subset U$, $((P_{\alpha+1} - P_\alpha)X)^*$ is also a subspace of a WCG Banach space since it is isomorphic to $(P^*_{\alpha+1} - P^*_\alpha)X^* \subset V$ and

$$\text{dens}\,((P_{\alpha+1} - P_\alpha)X)^* = \text{dens}\,(P^*_{\alpha+1} - P^*_\alpha)X^* < \aleph.$$

Now it is easy to see that the whole X admits a shrinking Markuševič basis (cf. [5]).

PROOF OF COROLLARY 1. See [11], Theorem 1.

PROOF OF COROLLARY 2. It may be shown by the methods discussed above that if $Y \subset X^*$ is a norm closed subspace of X^* then the P_α's from Lemma 4 can be built up such that they preserve Y (since $Y \subset X^* \subset V$). Then we may use the arguments of [5] where also a recent result of W. B. Johnson on w^* closed quasicomplements is used (see [5]).

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